Improved constraints on transit time distributions from argon 39: A maximum entropy approach

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Plan

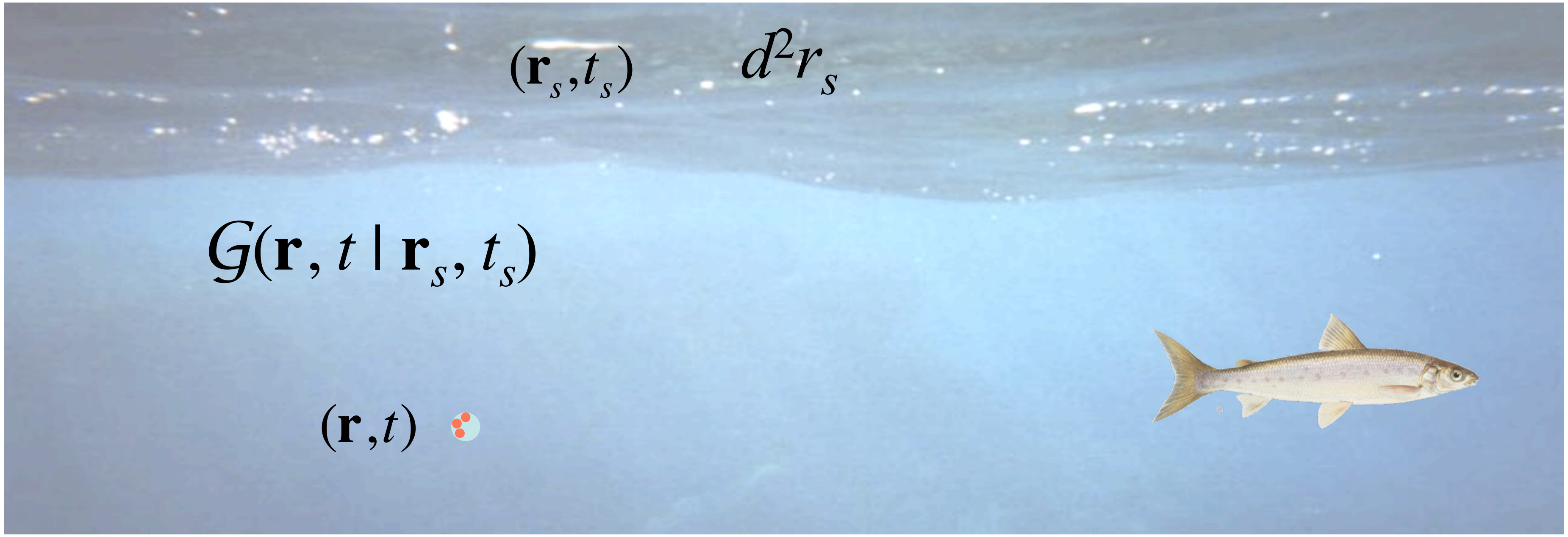
I. What we want to know: boundary propagator, $\mathcal{G}$

Deconvolve $\mathcal{G}$ from tracers: maximum entropy

II. The effect of Ar-39 on maximum-entropy estimates of the boundary propagator

- a simple 1d “toy” model
- the real 3d ocean

What we want to know:



What: Mass fraction $\mathcal{G}(\mathbf{r}, t \,|\, \mathbf{r}_s, t_s)\, d^2 r_s\, dt_s$ at $(\mathbf{r}, t)$ that had last surface (mixed-layer) contact at surface patch $d^2 r_s$ during $(t_s, t_s + dt_s)$

Why: $\mathcal{G}$ is a boundary propagator Green function for any passive species, e.g., anthropogenic CO_2

$$C_j^I(\mathbf{r}, t) = \int d^2 r_s \int_0^\infty dt_s\, \mathcal{G}(\mathbf{r}, t | \mathbf{r}_s, t_s)\, C_j^S(\mathbf{r}_s, t_s) e^{-\gamma_j (t - t_s)}$$

Interior concentration of j^{th} tracer

surf. conc. of j^{th} tracer

Radio. decay (^{39}Ar and ^{14}C)

Determining $\mathcal{G}$ from tracer data: a deconvolution problem

$$C_j^I(\mathbf{r}, t, [\mathcal{G}]) = \int d^2 r_s \int_0^\infty d\tau \, \mathcal{G}(\mathbf{r}, t | \mathbf{r}_s, t-\tau) \, C_j^S(\mathbf{r}_s, t-\tau) e^{-\gamma_j \tau}$$

Known, measured interior concentration

Unknown Green fn/Distribution

Known surface conc. attenuated by radio. decay if any

Approximate flow as cyclostationary and discretize surface location and transit time $\tau = t - t_s$

$$\mathcal{G}(\mathbf{r}, t | \mathbf{r}_s, t_s) \, d^2 r_s dt_s \rightarrow \mathcal{P}(s, n, m | \mathbf{r}, m_0) \equiv \mathcal{P}(s, \tau | \mathbf{r})$$

n = years since last contact
m = month of year of last contact
m_0 = month of year of OBS

$$C_j^I(\mathbf{r}, t, [\mathcal{P}]) = \sum_s \sum_\tau \mathcal{P}(s, \tau | \mathbf{r}) C_j^S(s, t-\tau) e^{-\gamma_j \tau}$$

A highly underdetermined problem:

How many unknown $\mathcal{P}$'s?

1800 surface locations (5°×5°)

53×12 discrete transit times (0 to 20,000 years)

$\Rightarrow \sim 10^6$ unknown $\mathcal{P}$'s

How many constraints?

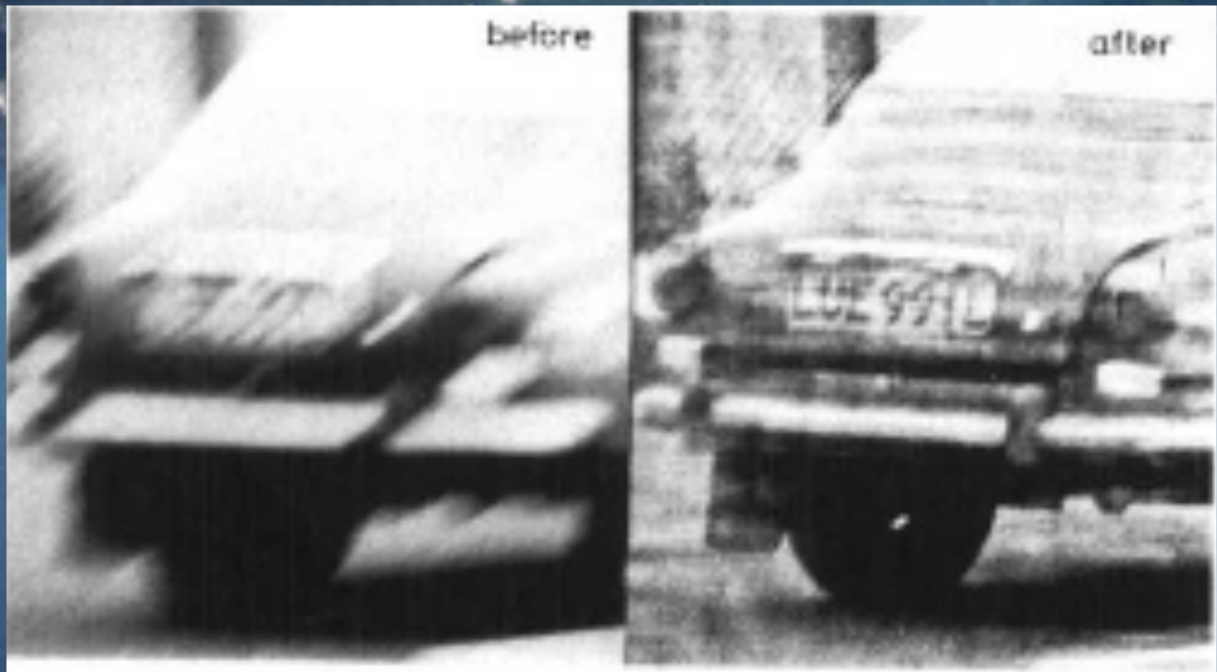
One convolution constraint for each trace species

$$C_j^I(\mathbf{r}, t, [\mathcal{P}]) = \sum_s \sum_\tau \mathcal{P}(s, \tau|\mathbf{r}) C_j^S(s, t - \tau) e^{-\gamma_j \tau}$$

6 or 7 tracers $\Rightarrow$ 6 or 7 constraints for 10^6 unknowns

Highly underconstrained!

Tracer-only estimates of $\mathcal{G}$
Maximum-entropy deconvolution



Holzer et al., JGR 2010; Holzer and Primeau, JGR 2010

Information entropy and maximum-entropy (ME) solution

Information entropy:
[Shannon, 1951]

$$\mathcal{S} = -\sum_{s,\tau} \mathcal{P} \log \frac{\mathcal{P}}{\mu} + \sum_{j=1}^{J} \lambda_j (C_j^I - C_j^I[\mathcal{P}])$$

OBS   from $\mathcal{P}$

ME solution:
[Jaynes, 1957]

$$\mathcal{P} = \frac{\mu}{Z} \exp\left(-\sum_j \lambda_j C_j^S(s, t-\tau) e^{-\gamma_j \tau} \right)$$

μ = "prior" distribution
λ_j = Lagrange multipliers
Z normalizes the distribution

Note that in the absence of constraints $\mathcal{P} = \mu$ (prior returned without data!)

Substitute ME soln. into J constraint equations and solve for the λ_j

The entropy has regularized the vastly underdetermined deconvolution to a problem of J equations in J unknowns!

Entropic probability density

Many $\mathcal{P}$ other than the ME solution $\mathcal{P}^*$ satisfy the data constraints. How likely are the alternatives?

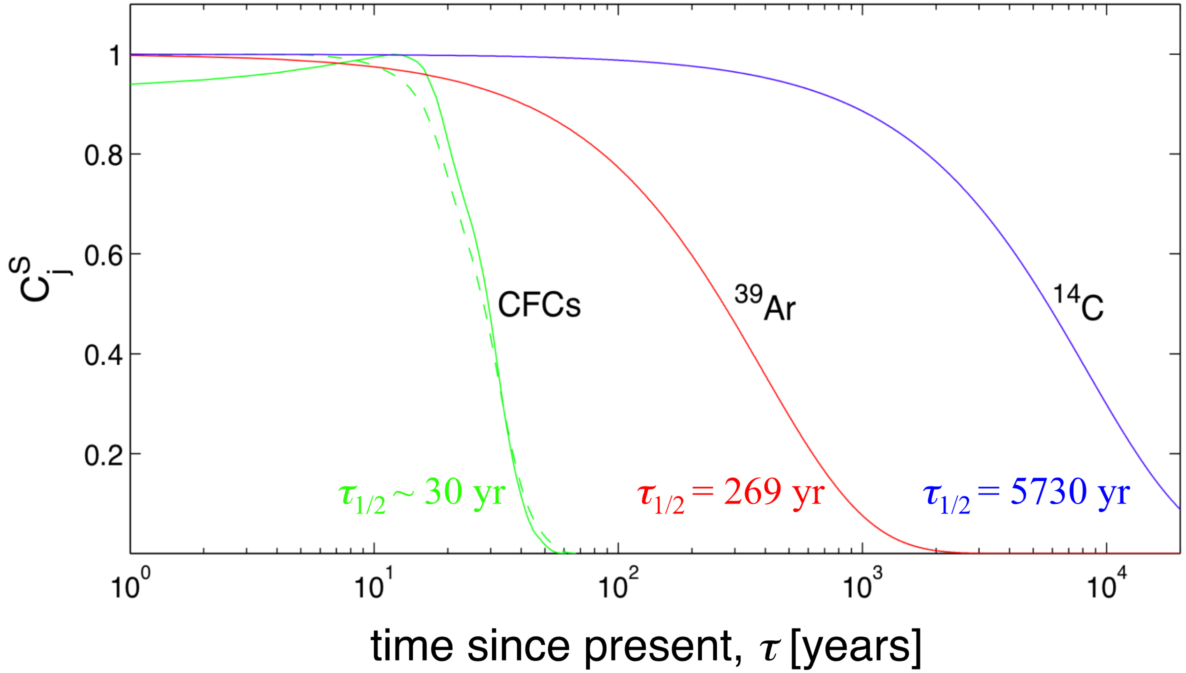
Extensions of the ME method [*Skilling and Gull*, 1991] assign a pdf to any $\mathcal{P}$:

$$p(\mathcal{P}) \propto \exp\left(-\frac{N}{2}\frac{\mathcal{S}(\mathcal{P})}{\mathcal{S}(\mathcal{P}^*)}\right)$$

We use this to quantify "**entropic uncertainties**."

A 1d toy model

Effective surface boundary conditions
$C^S(t - \tau)\exp(-\gamma\tau)$ for idealized 1d model



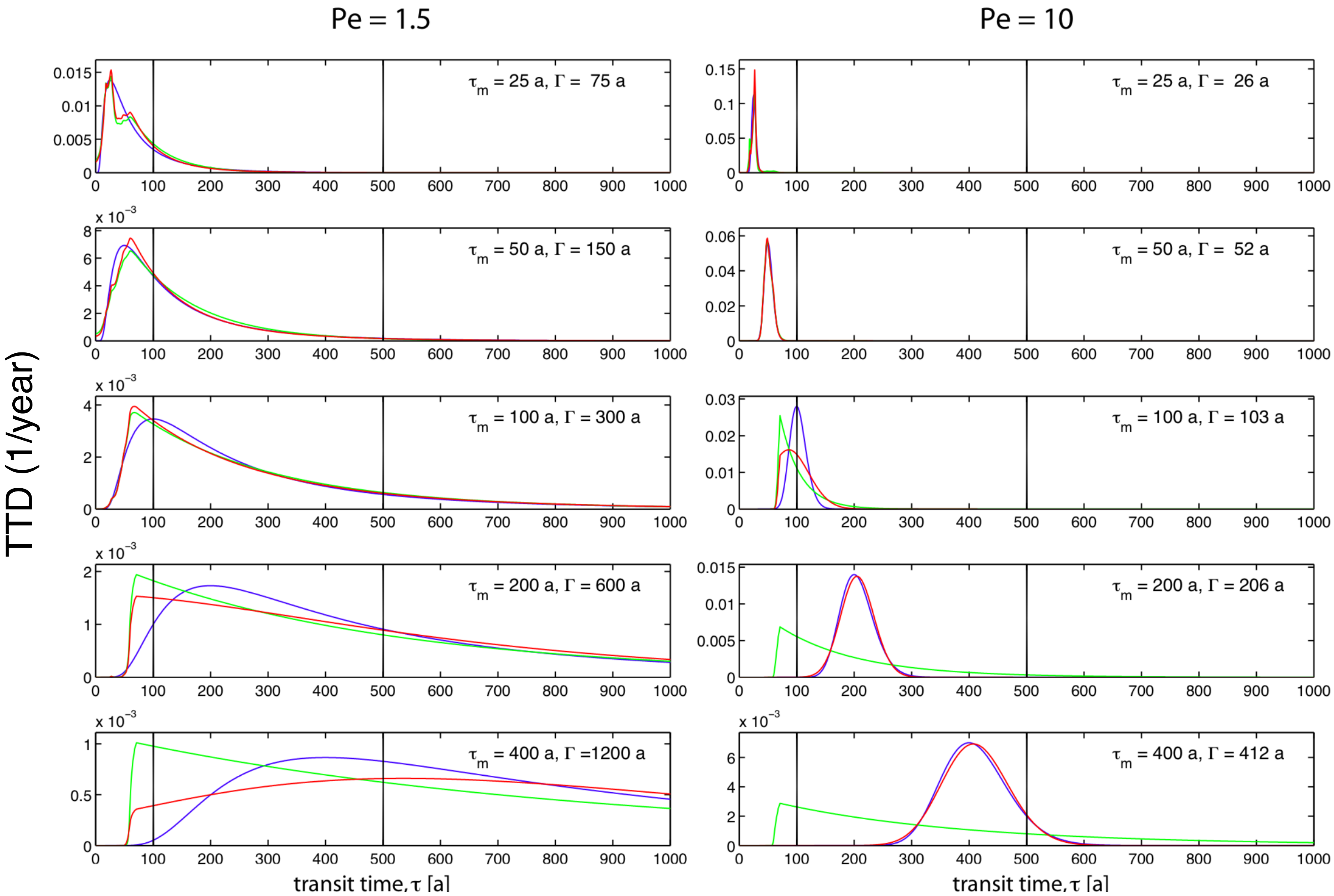
Assume simple parametric inverse Gaussian form for $\mathcal{G}$
mean age Γ, width Δ, Pe = Γ/Δ

$$\mathcal{G}(\hat{\tau}) = \frac{1}{\sqrt{4\pi\Delta^2\hat{\tau}^3}} e^{-\frac{1}{4}\left(\frac{\Gamma}{\Delta}\right)^2 \frac{(\hat{\tau}-1)^2}{\hat{\tau}}}$$

$$\hat{\tau} \equiv \tau/\Gamma$$

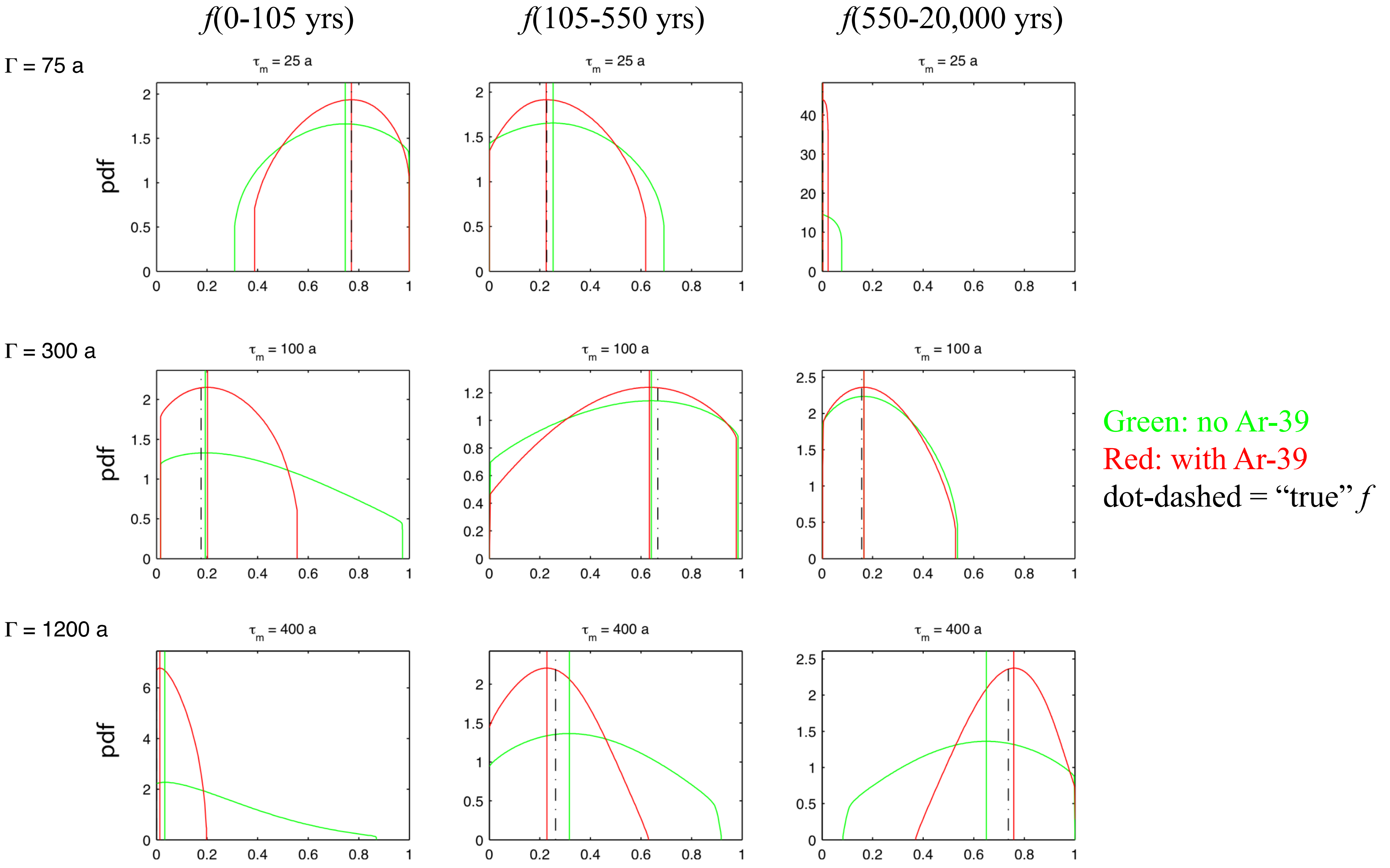
Effect of Ar-39 on ME TTDs for 1d toy model:

Generate synthetic CFC, Ar-39, C-14 interior data using analytic IG TTD and deconvolve using maximum entropy with and without Ar-39 (uniform prior)

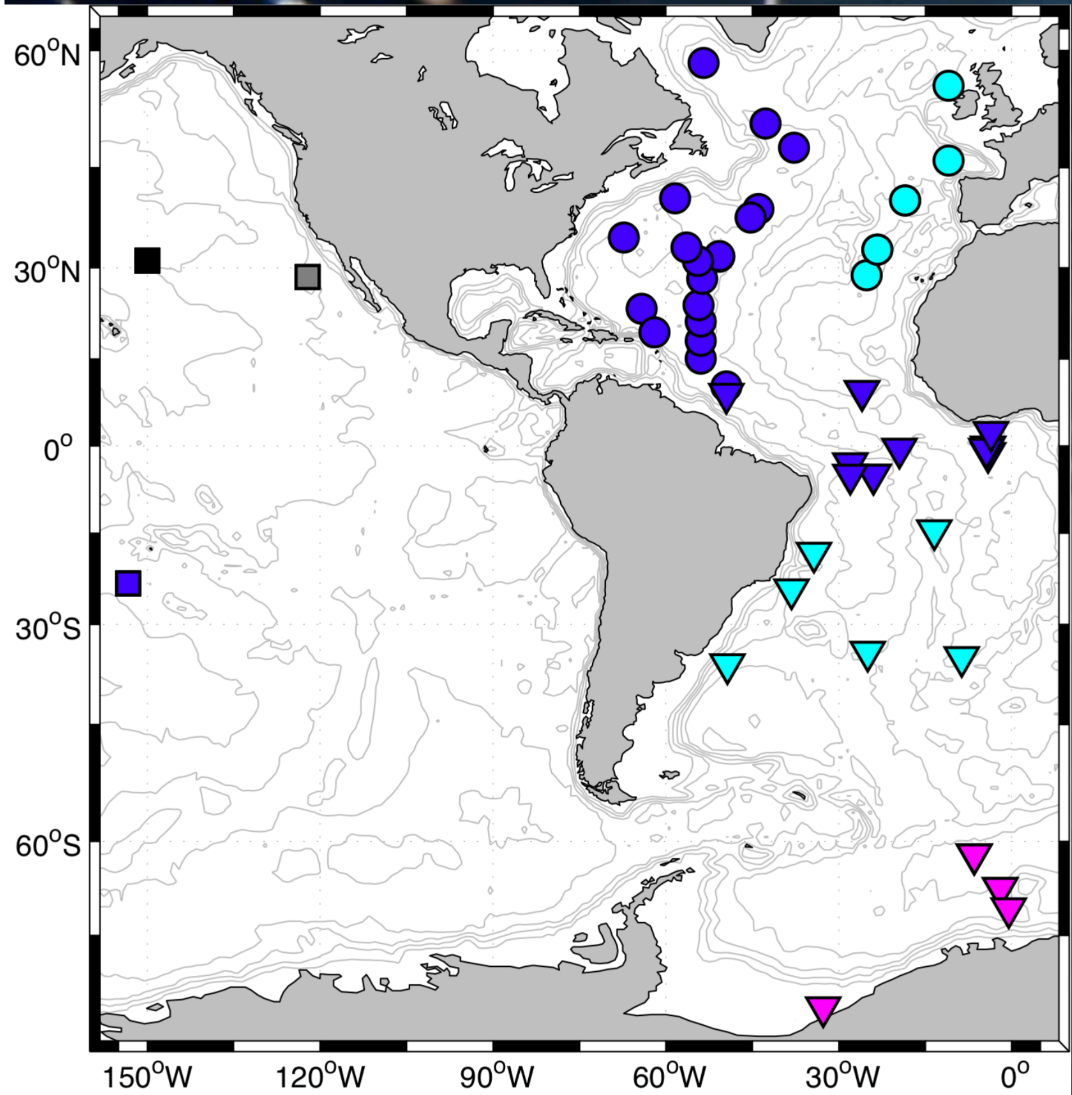


Blue: true TTD (used to generate interior data)
Green: ME soln. using CFCs and CF-14 only
Red: ME soln. using CFCs, C-14, and Ar-39

Effect of Ar-39 on entropic pdf of water-mass fractions in specified age band (1d toy model, Pe =1.5)



The real ocean



The locations of the ^{39}Ar measurements used (vertical profiles only at a few locations). Symbol shapes and colors label different regions. Measurements by Hugo Loosli [*Rodriguez, 1993*; *Loosli, 1893*].

Ar-39
C-14 (natural)
CFC-11
CFC-12
θ (potential temperature)
S (salinity)
PO_4* (conservative nutrient)

About 70 usable locations

Tracer data used

Cyclostationary: θ, S, $PO^*_4 = PO_4 + O_2/175$ [WOA05]

Transient: ^{39}Ar [*Loosli , Rodriguez '93*]

CFC-11, -12 [*Walker et al., 2000*]

^{14}C (natural) [GLODAP]

$$\chi_{\mathrm{Ar}} \equiv (^{39}\mathrm{Ar}/\mathrm{Ar})_{\mathrm{sample}}/(^{39}\mathrm{Ar}/\mathrm{Ar})_{\mathrm{atmos.}}$$

Because of rapid air-sea exchange of ^{39}Ar, assume mixed layer is saturated with ^{39}Ar, i.e., $\chi_{Ar}=1$ at the surface.

Because the boundary condition for ^{39}Ar is spatially uniform, ^{39}Ar helps only to constrain the transit-time dependence of $\mathcal{G}$ but not its surface-origin dependence

⇒ focus on global transit-time distribution TTD

TTD, ideal mean age, water-mass fraction

Transit-time distribution (TTD):

$$\mathcal{G}_{\mathrm{TTD}}(\mathbf{r},\tau) = \int d^2 r_s \, \mathcal{G}(\mathbf{r},t|\mathbf{r}_s,t-\tau) = \sum_s \mathcal{P}(s,\tau|\mathbf{r})$$

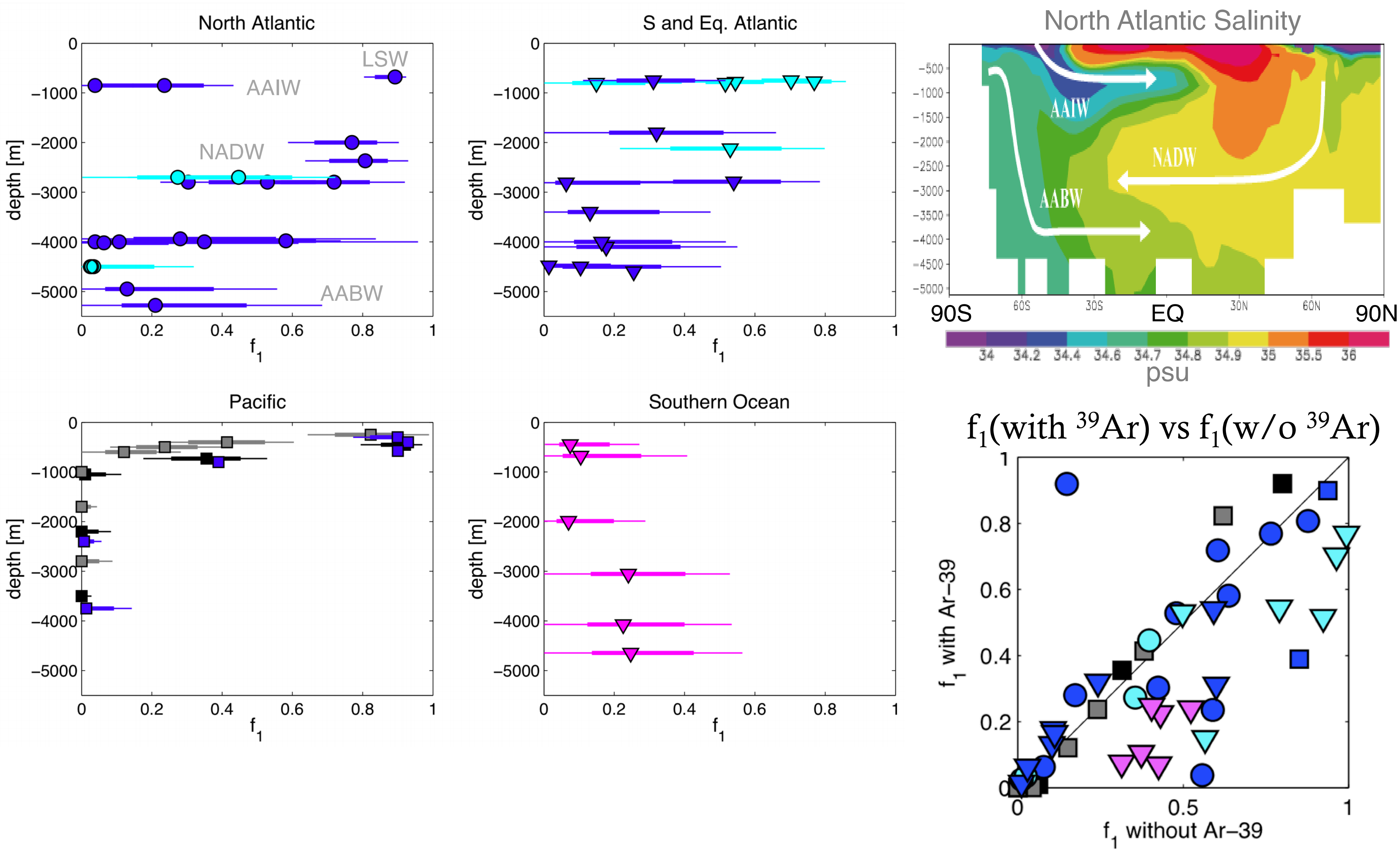
Ideal mean age, Γ :

$$\Gamma(\mathbf{r}) = \int_0^{\infty} \tau \, \mathcal{G}_{\mathrm{TTD}}(\mathbf{r},\tau)\, d\tau$$

Water-mass fraction, f, in given age band:

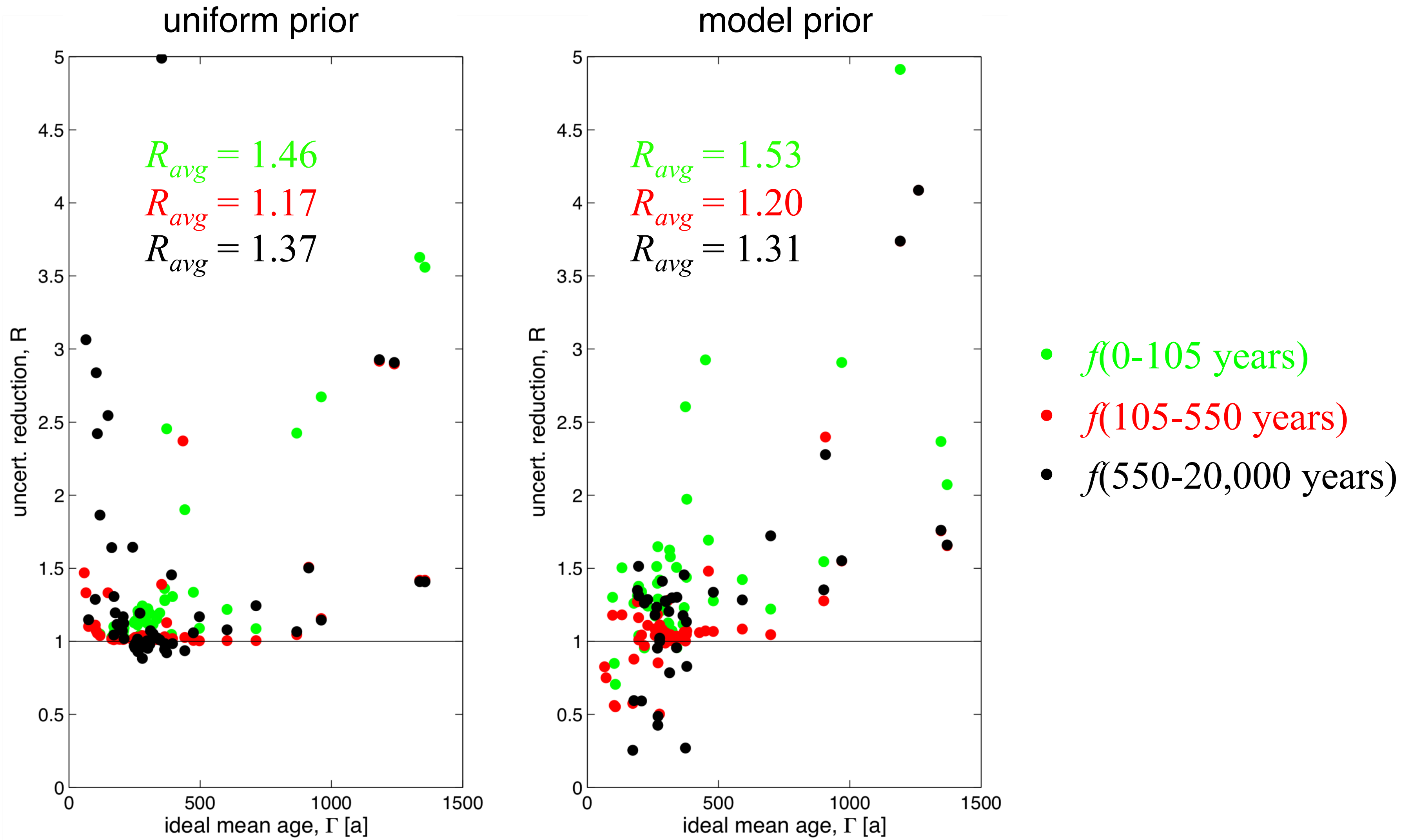
$$f(\mathbf{r},\tau_1,\tau_2) = \int_{\tau_1}^{\tau_2} \mathcal{G}_{\mathrm{TTD}}(\mathbf{r},\tau)\, d\tau$$

Water-mass fraction younger than 105 years (with Ar-39, AD prior)

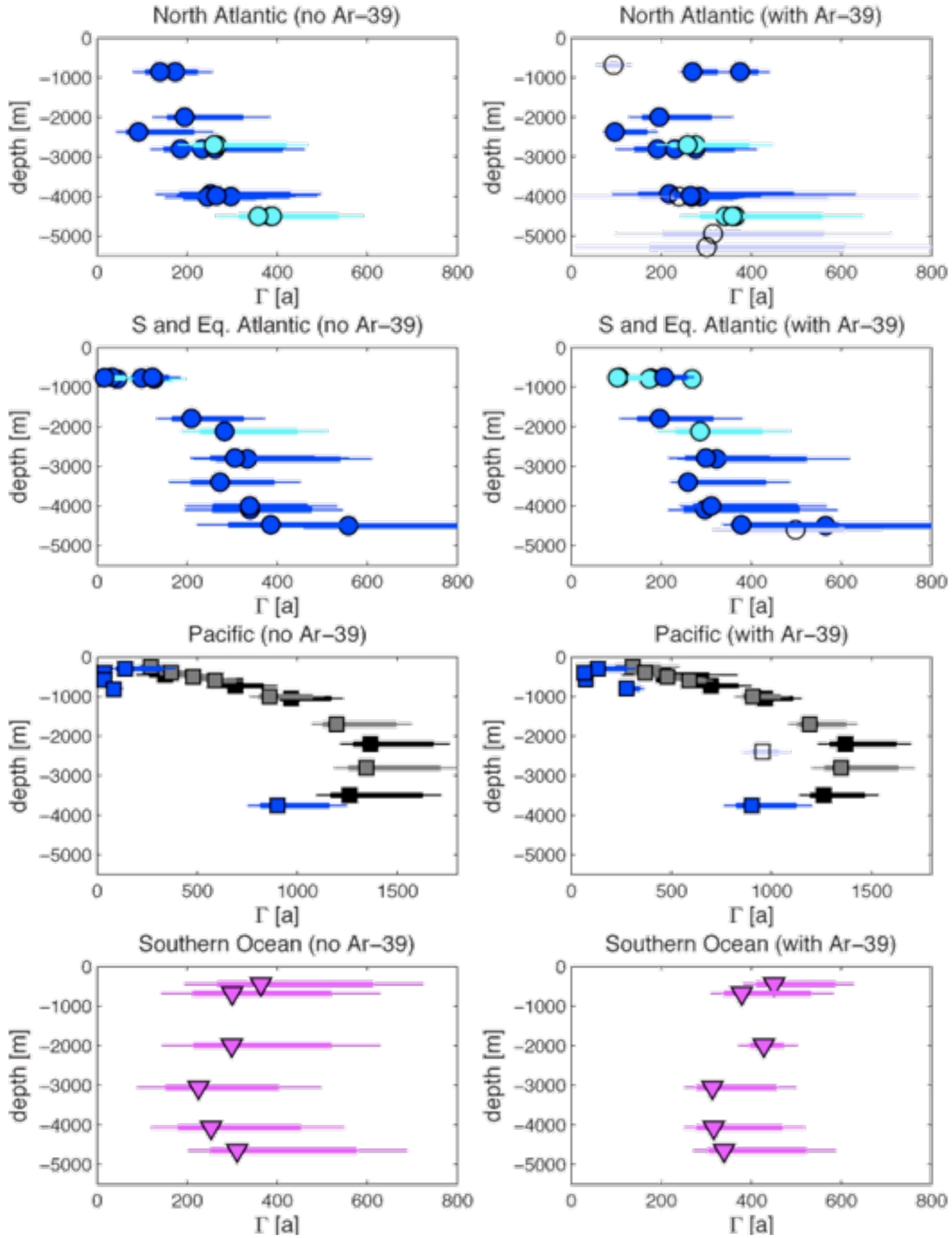


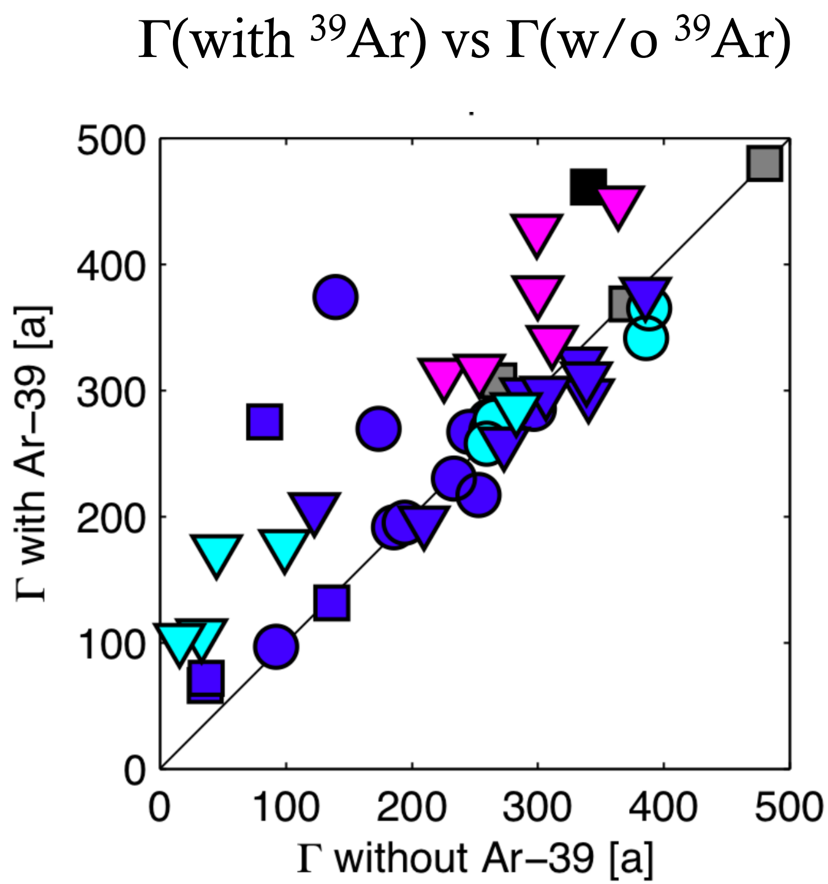
Uncertainty change due to inclusion of Ar-39 for water-mass fraction *f* in a given age band

uncertainty reduction $R \equiv \dfrac{\delta(\text{no }^{39}\text{Ar})}{\delta(\text{with }^{39}\text{Ar})}$



Ideal mean age, Γ

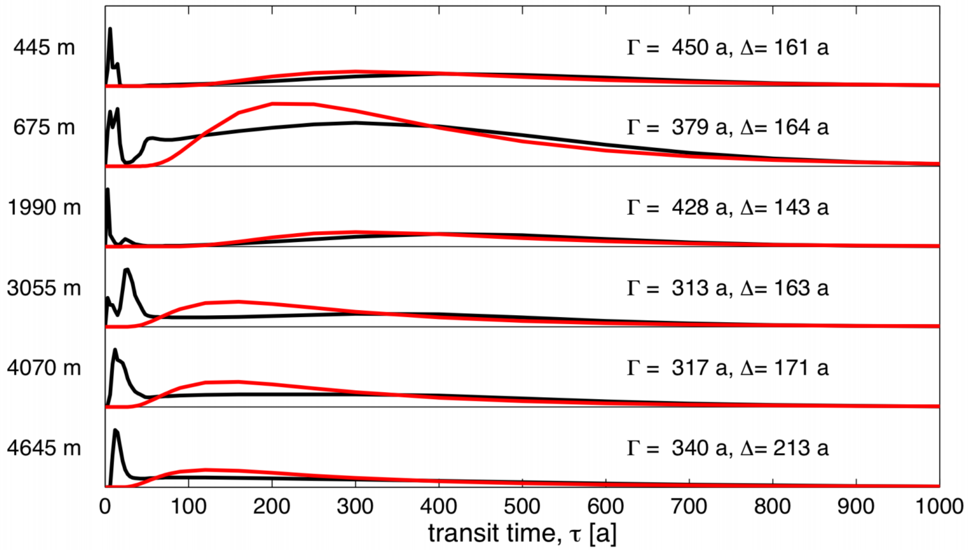




ME TTDs compared with IG:

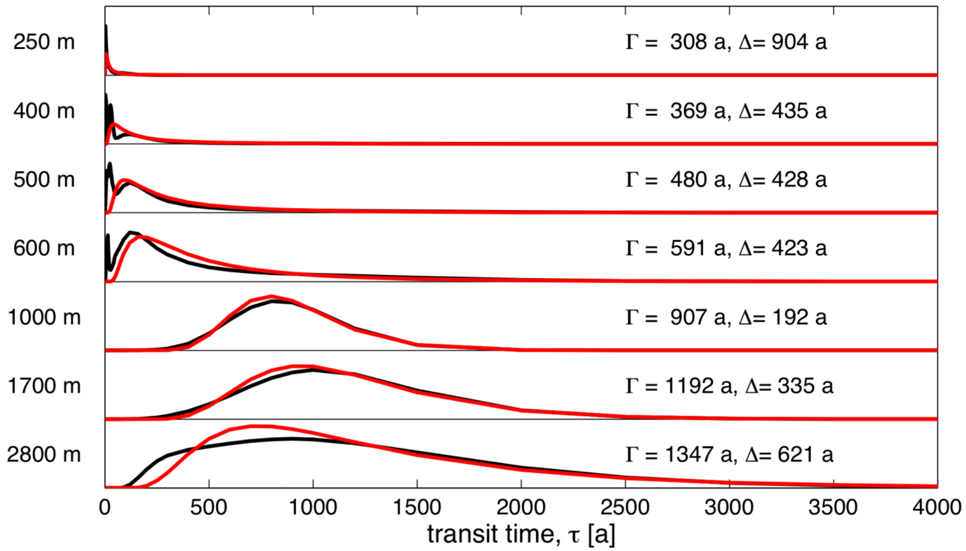
Southern Ocean

max. entropy, inverse gaussian



North Pacific (28.5 N, 122.2 W)

max. entropy, inverse gaussian



Conclusions:

1. **Including Ar-39 leads to systematic changes in ME-inferred transit-time distributions (TTDs): reduced young fraction, older ideal mean ages**

2. **Ar-39 tends to reduce the uncertainty (20-50% reductions) in the fraction of short and long transit times, with less effect on the 105-550 year band. Reductions not guaranteed everywhere.**

3. **Ar-39 gives gives ME inversion additional freedom (mode not stuck at early times for uniform prior).**

4. **In deep Pacific, Ar-39 plus C-14 allow determination of realistic TTDs where CFCs have not yet penetrated**

5. **TTD's generally do not have a simple unimodal inverse-Gaussian form often assumed by parametric inversions. Multimodality is important in the Southern Ocean, less in the North Pacific. Uncertainty in width/age ratio is likely larger than previously estimated. Need more than just mean age and width to characterize TTD.**

6. **Ar-39 plus T, S, PO_4*, CFC-11, CFC-12, C-14 provides sufficient constraints to make inversion insensitive to choice of prior (uniform vs model).**

Holzer, M. and F. W. Primeau, *J. Geophys. Research, 115*,
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http://web.maths.unsw.edu.au/~markholzer