r-mode instability and spin frequencies of compact stars

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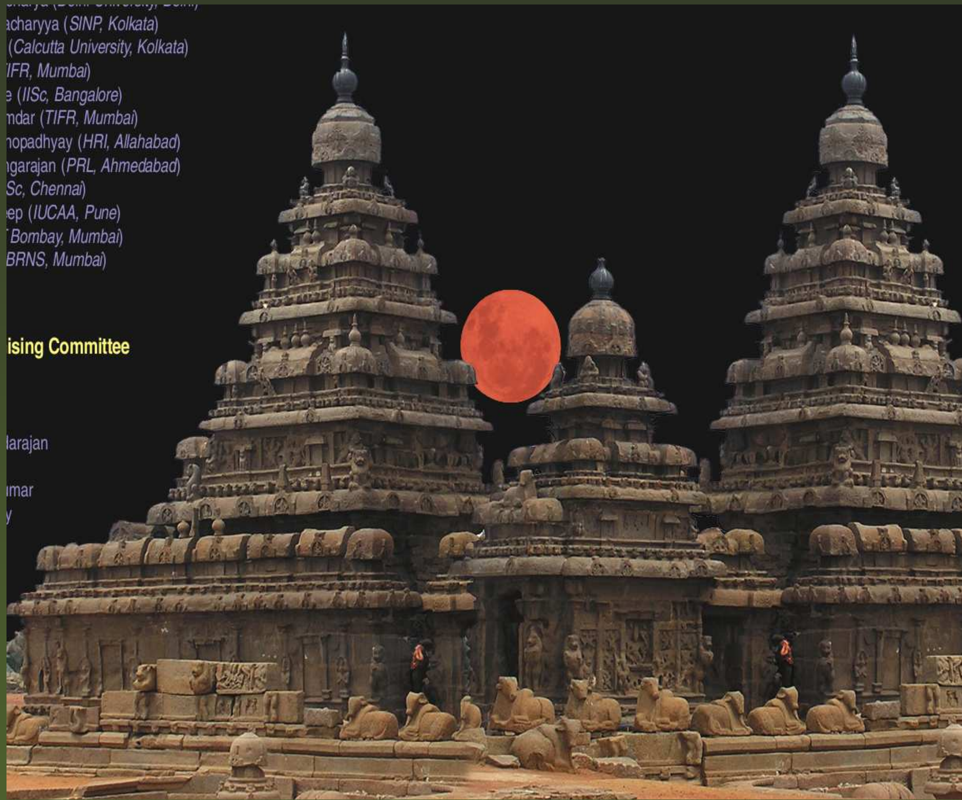

Collaborators on this work

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arXiv 0806.1005 (astro-ph)

Outline

- Stellar Oscillations: what information do they convey?

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- Neutron stars with/without quark matter: distinctions
- How fast can such compact stars spin?

Stellar Oscillations

Oscillation modes are classified by nature of restoring force

r-modes: Coriolis force ($\vec{\Omega} \times \vec{v}$) term in rotating stars

p-modes: Pressure fluctuations, convective instability

g-modes: Buoyancy (gravity) smooths out inhomogeneity

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f-modes (no radial node): Cepheid variables $\Rightarrow$ distance estimators

p-modes: used in helioseismography; verified standard solar model.

Oscillations of compact stars

$\boxed{\text{Perturbations trigger oscillations}}$

- Core-collapse: Neutron stars born in oscillatory state
- Crust-breaking and glitches lead to oscillations
- Interactions with companion/rapid mass-transfer
- Second collapse: phase transition to quark matter?

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stellar oscillations occur in a variety of astrophysical processes

1. What are the characteristic frequencies?
2. Information on structure of interior
3. Observations: spin rates, gravitational waves

Oscillatory solutions in non-rotating stars

Perturbed Euler equation (linearized)

$$\boxed{\partial_t(\delta\vec{v}) + \delta\vec{v}.\nabla\vec{v} = -\nabla\left(\frac{\delta P}{\rho} - \delta\Phi\right)}$$

Seek solutions of the form

$$\boxed{\delta\vec{v}_\perp = f(r)\vec{Y}_{lm}(\theta,\phi)\mathrm{e}^{i\omega_r t}; \quad \delta v_r \approx 0}$$

$\vec{Y}_{lm} \propto (\vec{r}\times\nabla)Y_{lm}$ are vector spherical harmonics

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Solutions can be classified by parity:

Spheroidal: transform as $(-1)^l$ (p,g-modes)

Toroidal: transform as $(-1)^{l+1}$ (r-modes)

Oscillatory solutions in rotating stars

Additional Coriolis force term: $\boxed{2(\vec{\Omega} \times \delta\vec{v})}$

In the fluid rest-frame, fluid displacement $\vec{\xi} = \int_0^t dt\, \delta\vec{v}$ obeys:

$$\boxed{-\omega_r^2 \vec{\xi} + 2i\omega_r (\vec{\Omega} \times \vec{\xi}) = -\nabla \left(\frac{\delta P}{\rho} - \delta\Phi \right)}$$

Employ the “Cowling approximation” for small Ω: $(\delta\Phi = 0)$

$$\boxed{\omega_r = \frac{2m\Omega}{l(l+1)}}$$

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To leading order in Ω (stellar rotation frequency), there is NO dependence on the equation of state.

r-mode: sensitivity to EoS

expand variables to $\mathcal{O}(\Omega^2)$: rotation modifies structure

$$\begin{aligned}
\rho(r,\theta) &= \rho_0(r) + \rho_2(r,\cos\theta)\frac{\Omega^2}{\pi G\bar{\rho}_0} + \mathcal{O}(\Omega^4) \\
\Phi(r,\theta) &= \Phi_0(r) + \Phi_2(r,\cos\theta)\frac{\Omega^2}{\pi G\bar{\rho}_0} + \mathcal{O}(\Omega^4)
\end{aligned}$$

The r-mode frequency becomes:

$$\boxed{\omega_r = \frac{2m\Omega}{l(l+1)}\left[1 - \kappa\frac{\Omega^2}{\pi G\bar{\rho}_0}\right]; \quad 0.1 \le \kappa \le 0.4}$$

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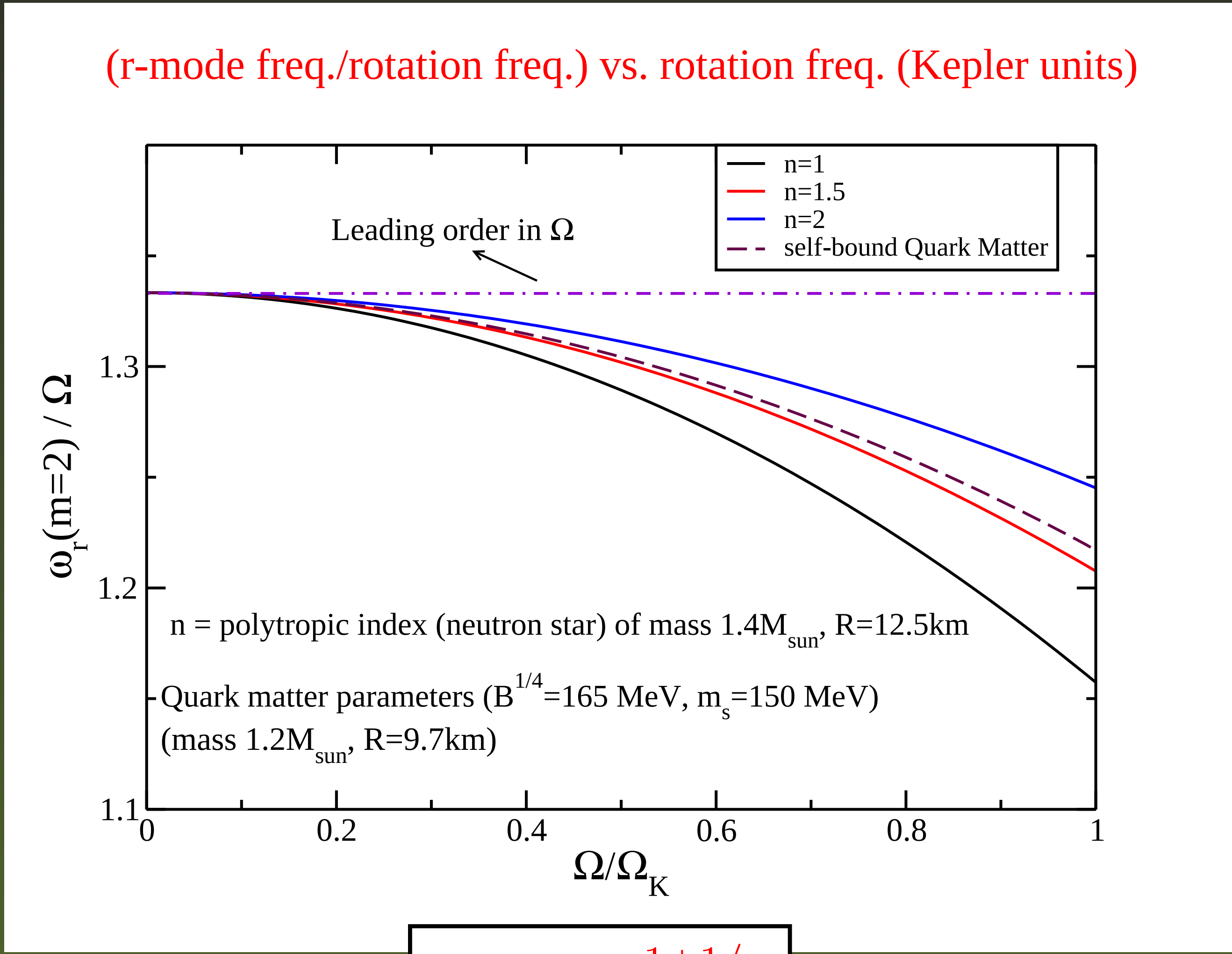
$\boxed{\kappa}$ depends on the density profile and radius of the star:

EoS dependence enters.

r-mode frequency



Neutron matter: Polytrope $\boxed{P = K\rho^{1+1/n}}$

Self-bound Quark matter: Bag model ($m_s \neq 0$)

Gravitational waves and r-mode instability

Contribution of gravitational waves to r-modes:

$$\left(\frac{dE}{dt}\right)_{\mathrm{GW}} \quad \propto \quad \boxed{-\sum_{m\geq 2}(\omega_r - m\Omega)^{2m+1}\omega_r \underbrace{|\delta J_{mm}|^2}_{current\ multipole}}$$

For $m \geq 2$, $\omega_r < m\Omega$, so the r-mode energy grows with gravitational wave emission, triggering the instability.

Viscosity and r-modes

Energy of r-mode is dissipated by bulk (ζ) and shear (η) viscosity

$$T_{ij} = \boxed{\zeta} \underbrace{\partial_k v_k}_{\sigma} \delta_{ij} + \boxed{\eta} \underbrace{\left(\partial_i v_j + \partial_j v_i - \frac{2}{3}\partial_k v_k \delta_{ij}\right)}_{\sigma_{ij}} - P\delta_{ij}$$

The energy contained in an r-mode is given by:

$$\boxed{E_r \propto R^4 \Omega^2 \int_0^R dr\, \rho_0(r) \left(\frac{r}{R}\right)^{2m+2} + \mathcal{O}(\Omega^4)}$$

..and is dissipated at the rate

$$\boxed{\frac{dE}{dt} = -\int (2\eta\delta\sigma^{ij}\delta\sigma_{ij} + \zeta\delta\sigma\delta\sigma) d^3r}$$

r-mode timescales

The timescale asociated to growth or dissipation (τ) is given by

$$\boxed{\frac{1}{\tau_i} = -\frac{1}{E}\left(\frac{dE}{dt}\right)_i \;; \quad i = \mathrm{GW}, \zeta, \eta}$$

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$\boxed{\tau_{\zeta,\eta} \gg \tau_{\mathrm{GW}}}$: r-modes will be effective in spinning down the star

$\boxed{\tau_{\zeta,\eta} \ll \tau_{\mathrm{GW}}}$: r-modes are quickly damped; star can spin rapidly!

r-mode Recap

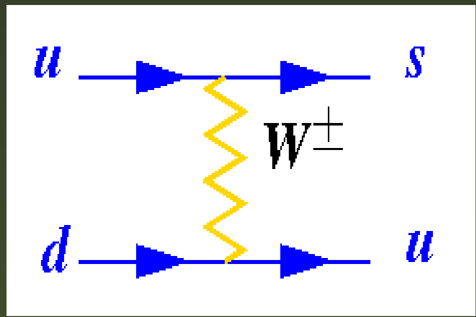
- r-mode oscillations is generic to all rotating stars
  (Coriolis force)

- The r-mode is unstable to gravitational-wave emission
  for all $m \geq 2$

- r-mode is damped by viscosity;
  exploit dependence of ζ, η on EoS

Bulk viscosity (ungapped quarks)

r-modes are low-frequency modes ($\sim$kHz) so only weak reactions are out of equilibrium $\boxed{d + u \leftrightarrow u + s}$:



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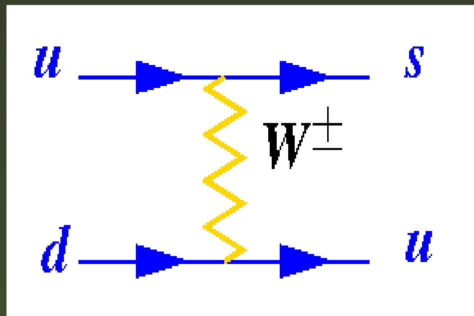

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For small perturbation amplitudes (J. Madsen, PRD 46, 3290 (1992))

$$\boxed{\zeta(\omega, T) = \frac{\alpha T^2}{\beta T^4 + \omega^2}}$$

Bulk viscosity of CFL (Alford et al., PRC 75, 055209 (2007))

- Lightest excitations in CFL are H-boson (superfluid phonon) and K (kaon)

$$\boxed{m_H = 0; \quad m_{K^0} \sim \frac{\Delta}{\mu_q}\sqrt{m_u(m_d + m_s)}}$$

- CFL Flavor re-equilibration : $\boxed{K^0 \to HH, K^0 H \to H}$

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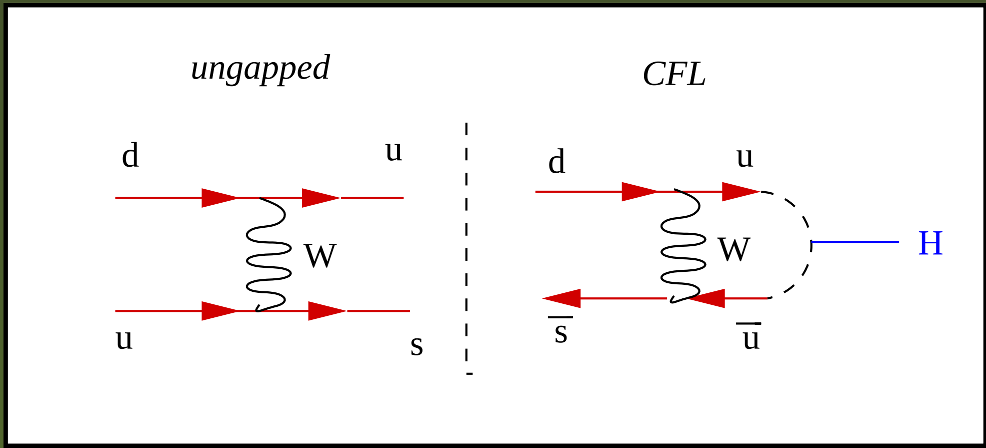


Shear viscosity

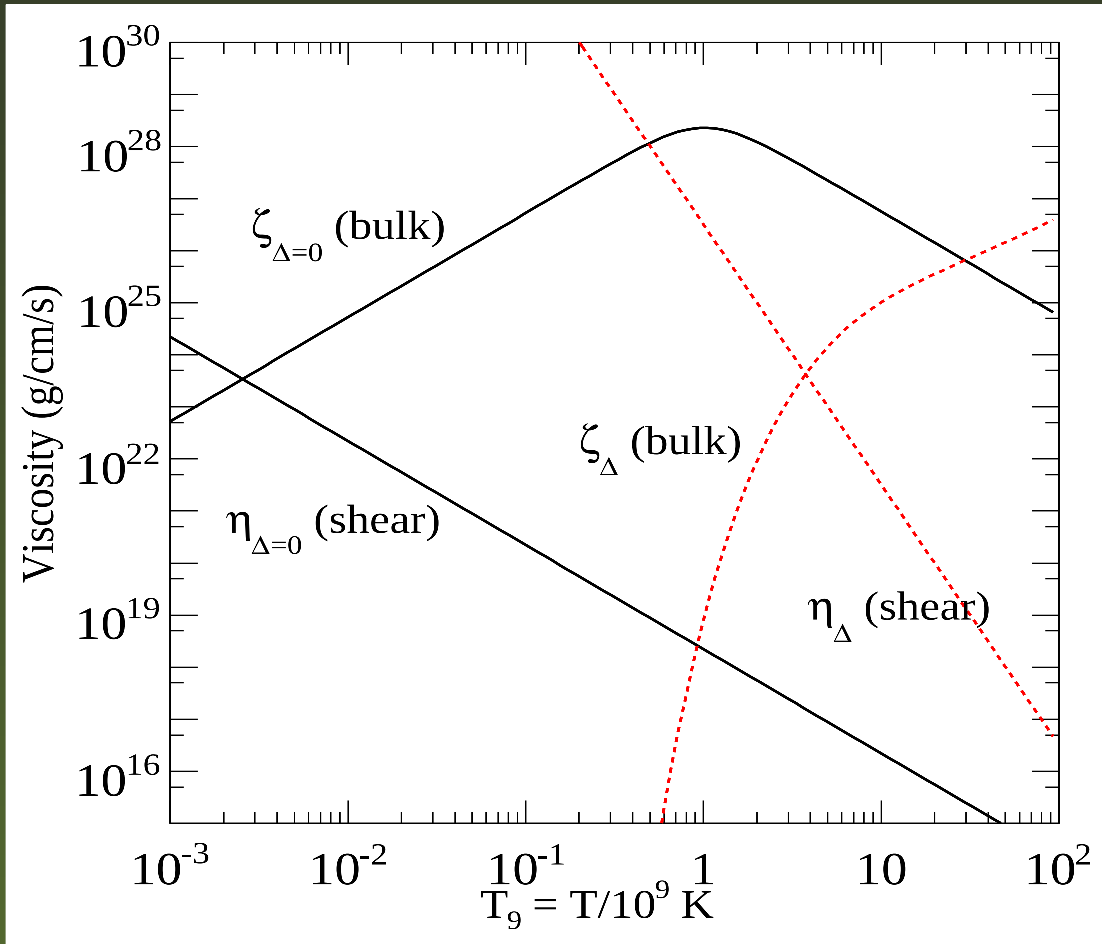
$\boxed{\eta}$ measures ease of momentum transport perpendicular to flow

- ungapped quark matter: η determined by qq scattering
- gapped (CFL) quark matter: η determined by small-angle phonon (H-boson) collisions $\boxed{\eta \approx 10^{-2} \frac{\mu^8}{T^5}}$

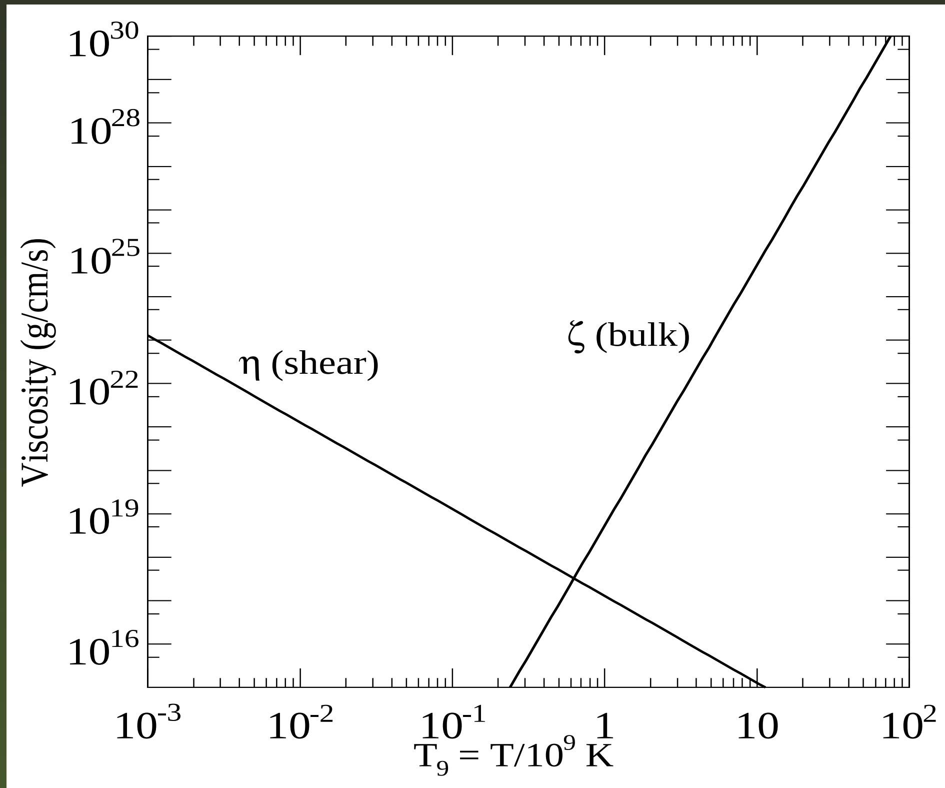
  (C. Manuel et al., JHEP 0509, 076 (2005))

Viscosities in Quark Matter

$\Delta = 0$ (normal quark matter); $\Delta > 0$ (gapped quark matter)



Viscosity of neutron matter

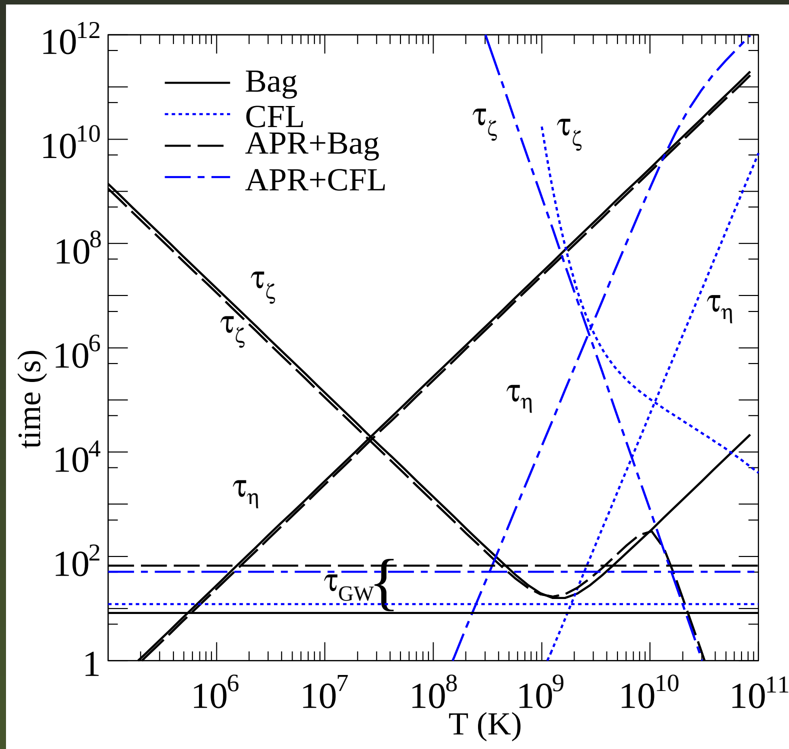


The bulk viscosity is controlled by the modified urca process:

$\boxed{n + n \to n + p + e^- + \bar{\nu}_e}$ $(\zeta \propto T^6)$

The shear viscosity is determined by nn scattering in non-superfluid matter ($\eta \propto 1/T^2$); by $ee, e\mu$ scattering in superfluid matter

r-mode damping timescales ($\Omega = \Omega_K$)



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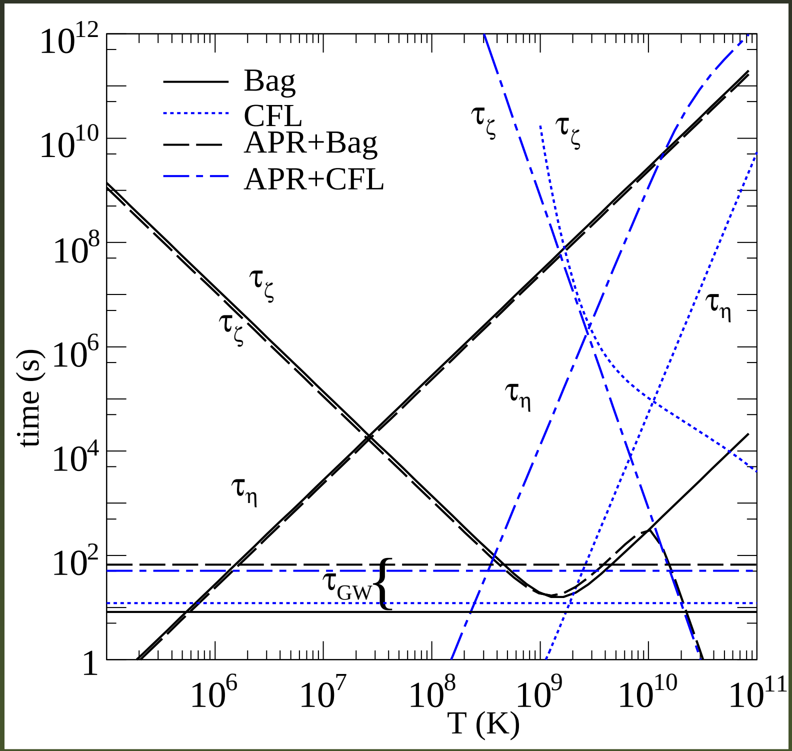

- Normal quark matter: Bulk viscosity damps r-mode instability in a wide range of T.
- CFL quark matter: r-mode is undamped in a narrow window (5×10^9K$\leq T \leq 5 \times 10^{10}$K)

Critical rotation frequency of compact stars

At the critical frequency Ω_c, fraction of energy dissipated/unit time exactly cancels against r-mode growth by gravitational wave emission.

$$\boxed{\frac{1}{\tau(\Omega_c)} = \left[\frac{1}{\tau_\zeta} + \frac{1}{\tau_\eta} + \frac{1}{\tau_{\mathrm{GW}}}\right](\Omega_c) = 0}$$

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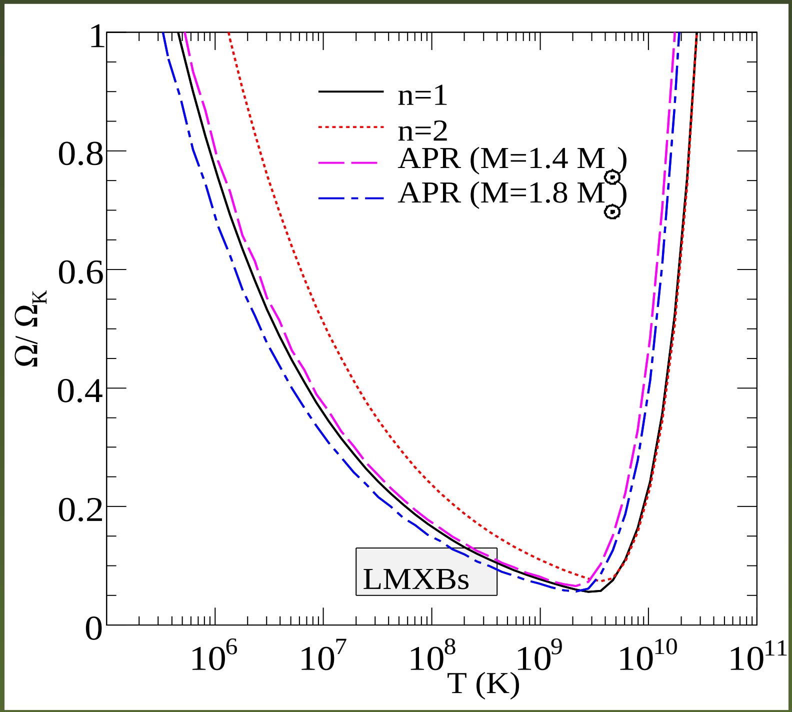

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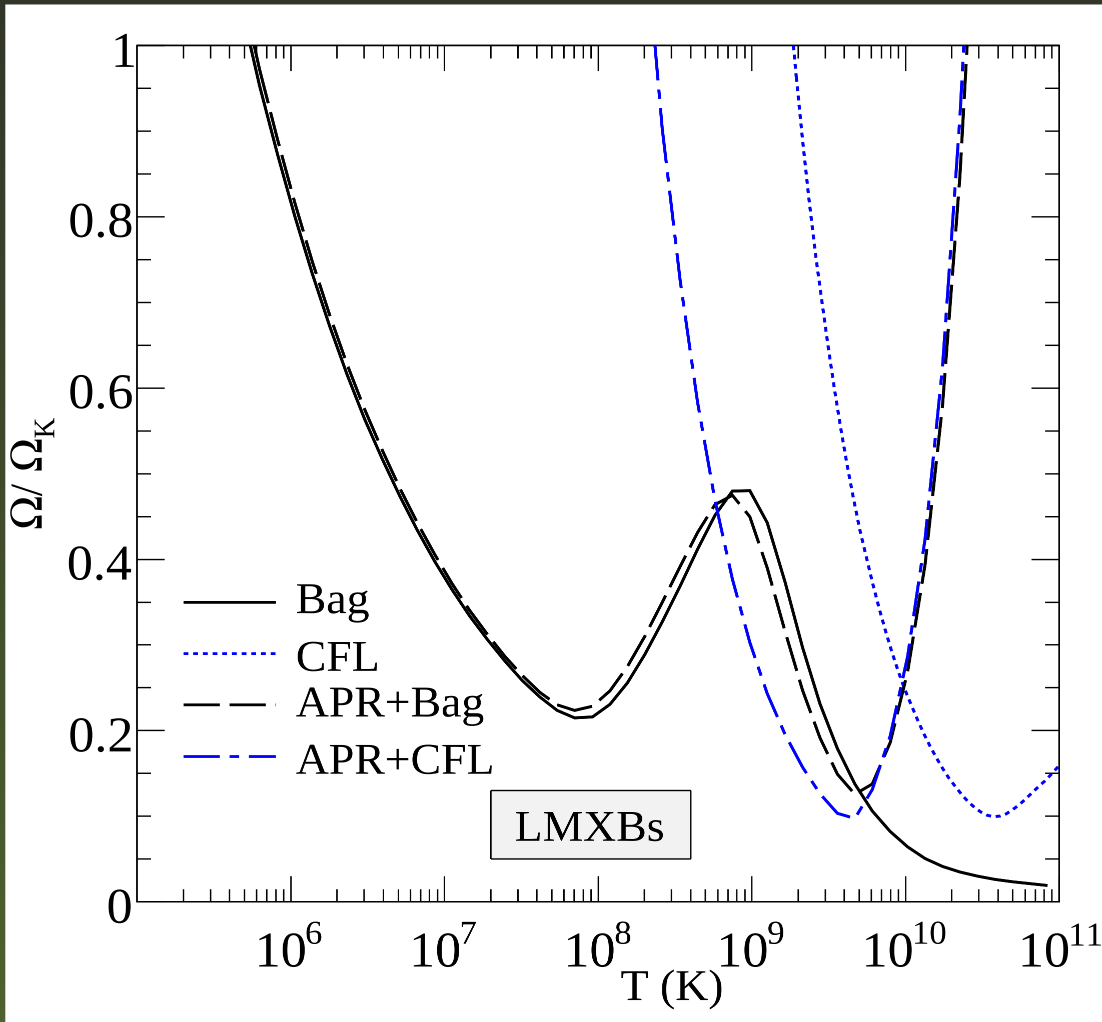

Results: Limits on rotation

- Neutron stars are stable against the r-mode instability at very high ($T \gtrsim 10^{10}$ K) or very low temperatures ($T \lesssim 10^{7}$ K).
  Neutron stars are spun down rapidly by the r-mode instability shortly after their birth at MeV temperatures.

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- Strange stars with non-superfluid quark matter display a stability window between $10^{8}\text{K} < T < 5 \times 10^{9}\text{K}$ where they can spin at a substantial fraction of the Kepler frequency.
- Strange stars in the CFL phase can spin at frequencies close to the Kepler limit even as they cool below 10^{10}K.
  LMXB's with quark matter can spin faster than observed limit

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- r-mode instability affects all rotating stars;
  determines how fast a compact star can spin
- Viscosity damps r-mode;
  new phases of dense matter change (ζ, η) and critical freq.
- CFL phase allows most rapid rotation frequencies;
  faster than observed LMXBs
- Studies of mixed & heterogenous phases more complicated

Recent works

- Viscosity and r-modes of 2SC, CSL phases

  B. Sa'd, arXiv:0806.3359

- Viscosity from urca ($d \rightarrow u + e^- + \bar{\nu}_e$) process in quark matter

  B. Sa'd, I. Shovkovy and D. Rischke, PRD 75 (2007) 125004

- Viscosity and r-modes of Kaon-condensed phases (n, p, K)

  D. Chatterjee and D. Bandyopadhyay, PRD 75 (2007) 123006

- Viscosity of Kaon-condensed CFL (moderate density)

  M. Alford, M. Braby & A. Schmitt, arXiv:0806.0285

- Mutual friction of the CFL phase (r-modes undamped)

  M. Mannarelli, C. Manuel & B. Sa'd, arXiv:0807.3264

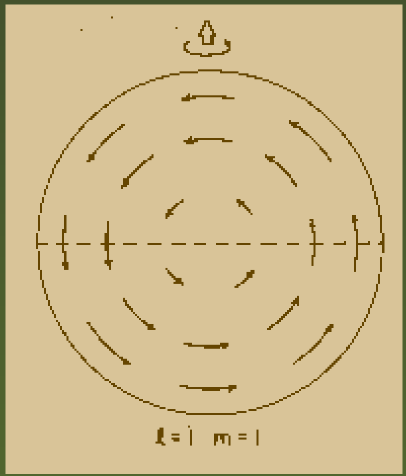
Visualizing r-modes

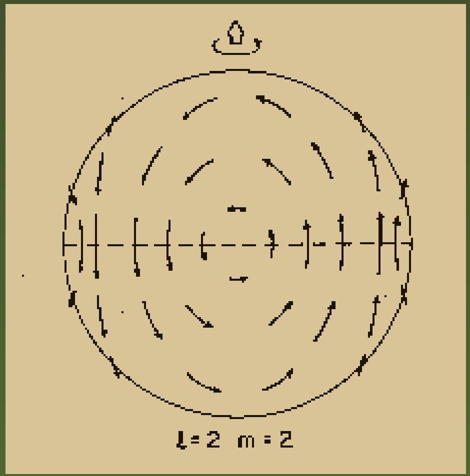
- The angular dependence of the flow (latitude dependence) is given by magnetic-type vector spherical harmonics:

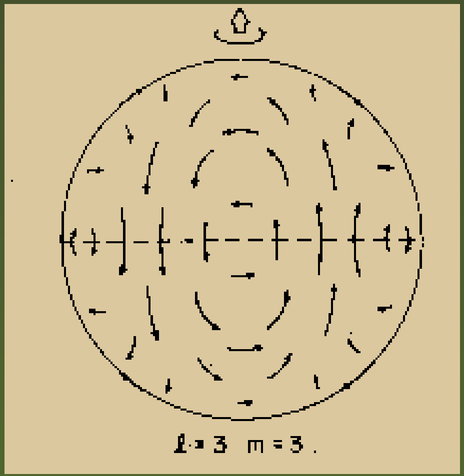
$$\boxed{\vec{Y}^{B}_{ll} = [l(l+1)]^{-1/2} r\nabla \times (r\nabla Y_{ll})}$$

- Flow of fluid element in r-mode conserves vorticity

$$\boxed{\frac{d}{dt}\left(\hat{e}_r.(\nabla \times \delta\vec{v}) + 2\hat{e}_r.\vec{\Omega}\right) = 0}$$

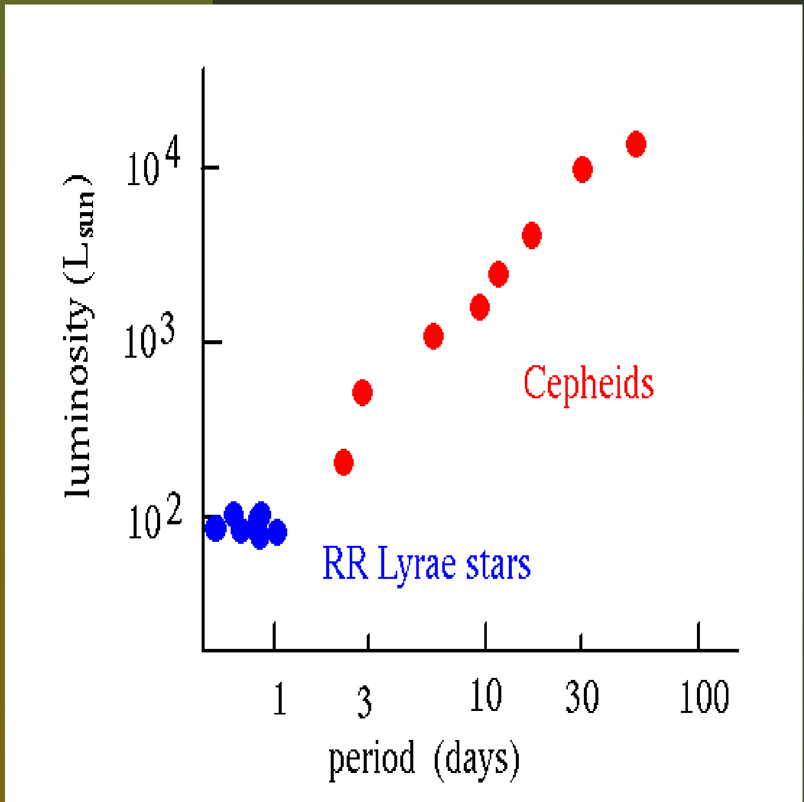






Cephoid variables

Henrietta Leavitt (1908)



Apparent brightness varies periodically

Absolute luminosity $\propto$ period of oscillation

Density oscillations ionize He-layer, change opacity

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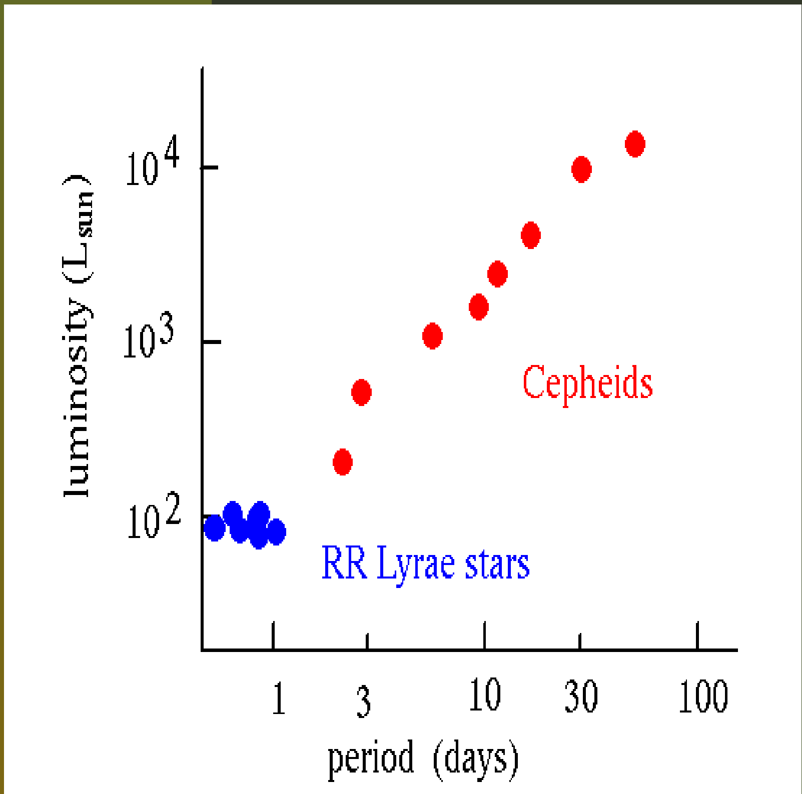

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Hertzsprung: Cepheid variables can be used as standard candles

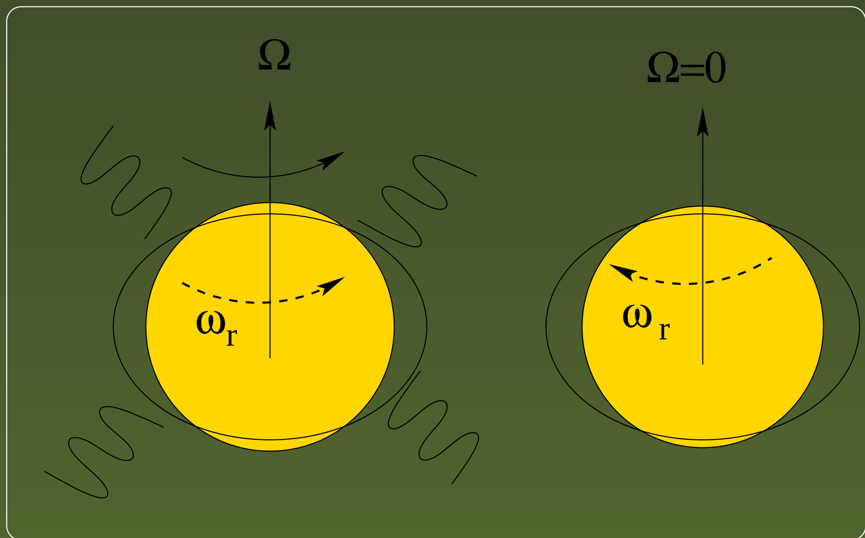
Hubble: Estimated distance to nearby galaxies

r-mode instability

An inertial observer measures an *effective* r-mode frequency

$$\boxed{\omega_r^{(in)} = \omega_r^{rot} - m\Omega = -\frac{(m-1)(m+2)}{(m+1)}\Omega}$$

For $m \geq 2$, a prograde mode in the inertial frame appears retrograde in the rotating frame



$$E^{rot} \uparrow = E^{in} \downarrow - \Omega\, J \downarrow$$

Fluid perturbation equations

(perturbed) Variables:

Energy density, Pressure $\boxed{\delta\rho, \delta P}$

Velocity $\boxed{\delta\vec{v}}$

Grav. Potential $\boxed{\delta\Phi}$

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Energy-momentum conservation $\boxed{\delta(\nabla_\mu T^{\mu\nu})=0}$

Poisson equation $\boxed{\delta(\nabla^2\Phi + 4\pi G\rho)=0}$

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Close the system: specify a barotropic Equation of State (EoS):

$\boxed{\text{Pressure vs. density } P = P(\rho)}$

Bulk viscosity

PdV dissipation due to chemical re-equilibration over compression cycle $\boxed{V(t) = V_0 + \mathrm{Re}[\delta V \mathrm{e}^{i\omega t}]; \qquad P(t) = P_0 + \mathrm{Re}[\delta P \mathrm{e}^{i\omega t}]}$:

phase lag between δV and δP due to finite equilibration rate (Γ)

$$\boxed{\zeta(\omega, T) = C(T) \frac{\Gamma}{\Gamma^2 + \omega^2}}$$

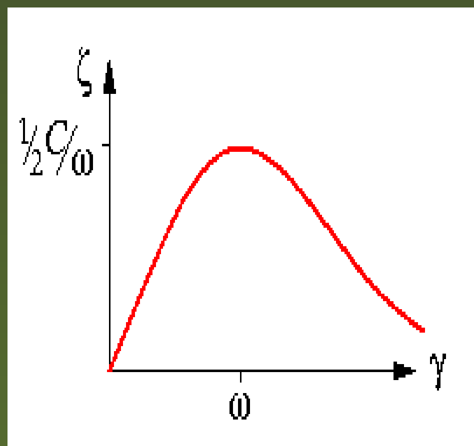
Bulk viscosity

PdV dissipation due to chemical re-equilibration over compression cycle $\boxed{\color{blue}{V(t) = V_0 + \mathrm{Re}[\delta V \mathrm{e}^{i\omega t}]}; \quad \color{red}{P(t) = P_0 + \mathrm{Re}[\delta P \mathrm{e}^{i\omega t}]}}$:

phase lag between δV and δP due to finite equilibration rate (Γ)

$$\boxed{\zeta(\omega, T) = C(T)\frac{\Gamma}{\Gamma^2 + \omega^2}}$$

Dissipation is maximum when frequency of mode is close to (any) equilibration rate in the fluid



Color superconducting phase

- weak attractive interaction between quarks at high density
  $\longrightarrow$ condensate of diquarks with color-flavor structure
- 3 massless quark flavors: Color-Flavor Locking (CFL)
  $\boxed{L \sim \langle q_i^a q_j^b \rangle_L; \quad R \sim \langle q_i^a q_j^b \rangle_R \sim \kappa \epsilon_{ijk} \epsilon^{abk}}$;
  $\boxed{SU(3)_c \times SU(3)_L \times SU(3)_R \times U(1)_B \rightarrow SU(3)_{c+L+R} \times Z_2}$

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Alford, Rajagopal and Wilczek (NPB 537, 443 (1999))

- Gapped excitations$\longrightarrow$ **9 quarks and 8 Higgsed gluons**

- Ungapped excitations$\longrightarrow$ **Nambu-Goldstone bosons**
A pseudoscalar (color-flavor) octet of mesons;
(like pseudoscalar flavor octet at μ_q=0): $\boxed{\pi, K, \eta}$